Free Products of Generalized RFD C*-algebras

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August 2, 2011

Abstract

If k is an infinite cardinal, we say a C*-algebra $\mathcal{A}$ is residually less than k dimensional, $R_{<k}D$, if the family of representations of $\mathcal{A}$ on Hilbert spaces of dimension less than k separates the points of $\mathcal{A}$. We give characterizations of this property, and we show that if $\{\mathcal{A}_i : i \in I\}$ is a family of $R_{<k}D$ algebras, then the free product $\underset{i\in I}{*}\mathcal{A}_i$ is $R_{<k}D$. If each $\mathcal{A}_i$ is unital, we give sufficient conditions, depending on the cardinal k, for the free product $\underset{i\in I}{*_{\mathbb{C}}}\mathcal{A}_i$ in the category of unital C*-algebras to be $R_{<k}D$. We also give a new characterization of RFD, in terms of a lifting property, for separable C*-algebras.

1 Introduction

A C*-algebra $\mathcal{A}$ is *residually finite dimensional* (RFD) if the collection of all finite-dimensional representations of $\mathcal{A}$ separate the points of $\mathcal{A}$; equivalently, if there is a direct sum of finite-dimensional representations of $\mathcal{A}$ with zero kernel. It is clear that every commutative C*-algebra is RFD. Man-Duen Choi [4] showed that free group C*-algebras are RFD. Ruy Exel and Terry Loring [6] proved that the free product of two RFD algebras is RFD. The class of RFD C*-algebras plays an important role in the theory of C*-algebras, e.g., [1], [2], [3], [4], [5], [6], [7], [11], [10].

In this paper we introduce a related notion. Suppose k is an infinite cardinal. We say that a C*-algebra $\mathcal{A}$ is *residually less than k-dimensional*, conveniently denoted by $R_{<k}D$, if the class of representations of $\mathcal{A}$ on Hilbert spaces of dimension less than k separates the points of $\mathcal{A}$; equivalently, if there is a direct sum of such representations that has zero kernel. Note that when $k = \aleph_0$, we have $R_{<k}D$ is the same as RFD. We give characterizations of $R_{<k}D$ algebras that show that the free product of an arbitrary collection of $R_{<k}D$ C*-algebras is $R_{<k}D$. We also give conditions that ensure that the free product (amalgamated over $\mathbb{C}$) of unital C*-algebras in the category of unital C*-algebras is $R_{<k}D$; this always happens when each of the algebras has a one-dimensional unital representation.

The proofs rely on a simple result (Lemma 1) and results of the author [8], [9] on approximate unitary equivalence and approximate summands of nonseparable representations of nonseparable C*-algebras.

Suppose k and m are infinite cardinals. We say that a C*-algebra $\mathcal{A}$ is *m-generated* if is generated by a set with cardinality at most m. For each cardinal s, we let H_s be a Hilbert space whose dimension is s. If $\pi : \mathcal{A} \to B(H)$ is a $*$-homomorphism, we say that the *dimension* of π is $\dim \pi = \dim H$. We define $\mathrm{Rep}_k(\mathcal{A})$ to be the set of all representations $\pi : \mathcal{A} \to B(H_s)$ for some $s < k$.

If $\mathcal{A}$ is a C*-algebra, then $\mathcal{A}^+$ denotes the C*-algebra obtained by adding a unit to $\mathcal{A}$ (that is different from the unit in $\mathcal{A}$ if $\mathcal{A}$ is unital).

We end this section with our key lemma. Suppose H is a Hilbert space and P is a projection in $B(H)$. We define $\mathcal{M}_P = PB(H)P$. Then $\mathcal{M}_P$ is a unital C*-algebra, but the unit is P, not 1. However, $\mathcal{M}_P$ is a C*-subalgebra of $B(H)$. A unitary element of $\mathcal{M}_P$ is an operator $U \in B(H)$ such that $UU^* = U^*U = P$, and is the direct sum of a unitary operator on $P(H)$ with 0 on $P(H)^\perp$. If $P \neq 1$, a unitary operator in $\mathcal{M}_P$ is never unitary in $B(H)$.

We use the symbol $*$-SOT to denote the $*$-strong operator topology.

Lemma 1 *Suppose $\{P_\alpha\}$ is a net of projections in $B(H)$ such that $P_\alpha \to 1$ ($*$-SOT) and let*

$$\mathcal{B} = \left\{ \{T_\alpha\} \in \prod_\alpha \mathcal{M}_{P_\alpha} : \exists T \in B(H), T_\alpha \to T \ (\ *\text{-}SOT \ ) \right\},$$

and

$$\mathcal{J} = \{\{T_\alpha\} \in \mathcal{B} : T_\alpha \to 0 \ (\ *\text{-}SOT \ )\},$$

and define $\pi : \mathcal{B} \to B(H)$ by

$$\pi(\{T_\alpha\}) = (*\text{-}SOT)\text{-}\lim_\alpha T_\alpha.$$

Then

1. *$\mathcal{B}$ is a unital C*-algebra,*
2. *$\mathcal{J}$ is a closed two-sided ideal in $\mathcal{B}$,*
3. *If $T \in \mathcal{B}$, then $\pi(\{P_\alpha T P_\alpha\}) = T$,*
4. *π is a unital surjective $*$-homomorphism*
5. *If $U \in B(H)$ is unitary, then there is a unitary $\{U_\alpha\} \in \mathcal{B}$ such that*

$$\pi(\{U_\alpha\}) = U.$$

Proof. Statements (1)-(4) are easily proved. To prove (5), note that if $U \in B(H)$ is unitary, then there is an $A = A^* \in B(H)$ such that $U = e^{iA}$. We can easily choose $A_\alpha = A_\alpha^*$ for each α so that $\pi(\{A_\alpha\}) = A$. Thus, if $U_\alpha = e^{iA_\alpha}$ (in $\mathcal{M}_{P_\alpha}$), then $\{U_\alpha\}$ is unitary in $\mathcal{B}$ and $\pi(\{U_\alpha\}) = U$. ■

Here is a simple application that gives the flavor of our results.

Corollary 2 *Every free group is RFD.*

Proof. Suppose $\mathbb{F}$ is a free group and $\mathcal{A} = \mathcal{C}^*(\mathbb{F}) = C^*(\{U_g : g \in \mathbb{F}\})$. Choose a Hilbert space H and a faithful representation $\rho : \mathcal{A} \to B(H)$. Choose a net $\{P_\alpha\}$ of finite-rank projections such that $P_\alpha \to 1$ ($*$-SOT). Applying Lemma 1 we have, for each $g \in \mathbb{F}$, we can find a unitary element $\{U_{g,\alpha}\}$ in $\mathcal{B}$ so that $\pi(\{U_{g,\alpha}\}) = U_g$. For each α, we have a unitary group representation $\sigma_\alpha : \mathbb{F} \to \mathcal{M}_{P_\alpha}$ defined by

$$\sigma_\alpha(g) = U_{g,\alpha}.$$

By the definition of $C^*(\mathbb{F})$, there is a $*$-homomorphism $\tau_\alpha : \mathcal{A} \to \mathcal{M}_\alpha$ such that $\tau_\alpha(U_g) = U_{g,\alpha}$. It follows that $\tau : \mathcal{A} \to \mathcal{B}$ define by $\tau(U_g) = \{U_{g,a}\}$ is a $*$-homomorphism such that $\pi \circ \tau = \rho$. Hence the direct sum of the τ_α's is faithful, which shows that $\mathcal{A}$ is RFD. ■

The following corollary is from [3, Exercise 7.1.4].

Corollary 3 *Every C*-algebra is a $*$-homomorphic image of an RFD C*-algebra.*

Proof. Suppose $\mathcal{A}$ is a C*-algebra. We can assume that $A \subseteq B(H)$ for some Hilbert space H. Choose a net $\{P_\alpha\}$ of finite-rank projections converging $*$-strongly to 1, and let $\mathcal{B}, \mathcal{J}$ and π be as in Lemma 1. Then $\mathcal{B}$, and thus $\pi^{-1}(\mathcal{A})$, is RFD and $\pi\left(\pi^{-1}(\mathcal{A})\right) = \mathcal{A}$. ■

2 $R_{<k}D$ Algebras

We now prove our main results on $R_{<k}D$ C*-algebras. The following two lemmas contain the key tools.

Lemma 4 *Suppose $\aleph_0 \leq k \leq m$, and $\mathcal{A}$ is $R_{<k}D$ and m-generated. Then*

1. We can write $H_m = \sum_{\lambda\in\Lambda}^{\oplus} X_\lambda$ with $Card\Lambda = m$, and such that, for every $\lambda \in \Lambda$, $\dim X_\lambda < k$ and there is a unital representation $\pi_\lambda : \mathcal{A}^+ \to B(X_\lambda)$ such that the representation $\pi : \mathcal{A}^+ \to B(H_m)$ defined by $\pi = \sum^{\oplus} \pi_\lambda$ is faithful. Moreover, this can be done so that, for each $\lambda_0 \in \Lambda$, we have $Card(\{\lambda \in \Lambda : \pi_\lambda \approx \pi_{\lambda_0}\}) = m$.

2. It is possible to choose the decomposition in (1) so that, for each cardinal $s < k$, there is a $\lambda \in \Lambda$ such that $\dim X_\lambda = s$.

Proof. Since $\mathcal{A}$ is $R_{<k}D$, there is a direct sum of representations in $\mathrm{Rep}_k(\mathcal{A})$ whose direct sum is faithful. Suppose D is a generating set for $\mathcal{A}$ and $Card(D) \leq m$. We can replace D by the $*$-algebra over $\mathbb{Q} + i\mathbb{Q}$ generated by D without making the cardinality exceed m. For each $a \in D$ we can choose a direct sum of countably many summands from our faithful direct sum that preserves the norm of a. Hence, by choosing $\aleph_0 Car(D)$ summands, we get a direct sum that is isometric on D and thus isometric on $\mathcal{A}$. Since $\aleph_0 Car(D) \leq m$. we can replace this last direct sum with a direct sum of m copies of itself and get a direct sum on a Hilbert space with dimension m. We can replace this Hilbert space with H_m and get a decomposition as in (1). to get (2) note that, since $\mathcal{A}^+$ has a unital one-dimensional representation, we know that, for every cardinal $s < k$. there is a representation of $\mathcal{A}^+$ of dimension s. If we take one such representation for each $s < k$ and take a direct sum of m copies of all of them, we get a representation that has has dimension at most m, so we add this as a summand to the representation we constructed satisfying (1). ■

Lemma 5 *Suppose $\mathcal{A}$ is a C^*-algebra and $k \leq m$ are infinite cardinals and D is a generating set for $\mathcal{A}$. Suppose we can write $H_m = \sum_{\lambda \in \Lambda}^{\oplus} X_\lambda$ and $\pi = \sum^{\oplus} \pi_\lambda$ as in part (1) of Lemma 4. If $\rho : \mathcal{A}^+ \to B(H_m)$ is a unital representation, then, for every $\varepsilon > 0$, every finite subset $W \subseteq \mathcal{D}$ and every finite subset $E \subseteq H_m$, there is a finite subset $F \subseteq \Lambda$, such that, for every finite set G with $F \subseteq G \subseteq \Lambda$, if Q_G is the orthogonal projection onto $\sum_{\lambda \in G}^{\oplus} X_\lambda$, then there is a unitary $U \in Q_G B(H_m) Q_G$ such that, for every $a \in W$ and $e \in E$, we have*

$$\|[\rho(a) - U_G^* \pi(a) U_G] e\| = \left\| \left[\rho(a) - U_G^* \left(\sum_{\lambda \in G} \pi_\lambda \right)(a) U_G \right] e \right\| < \varepsilon.$$

Proof. It follows that if $a \in \mathcal{A}$ and $a \neq 0$, then $rank\pi(a) = m = rank(\pi \oplus \rho)(a)$. Hence, by [8], π is approximately unitarily equivalent to $\pi \oplus \rho$. However, by [9], ρ is a point-$*$-SOT limit of representations unitarily to ρ. Hence there is a net $\{U_\alpha\}$ of unitary operators in $B(H_m)$ such that, for every $a \in \mathcal{A}$,

$$(*\text{-SOT}) \lim_\alpha U_\alpha^* \pi(a) U_\alpha = \rho(a).$$

However, the net $\{Q_F : F \subseteq \Lambda,\ F \text{ is finite}\}$ is a net of projections converging $*$-strongly to 1. Hence, by Lemma 1, each U_α is a $*$-SOT limit of unitaries in the union of $Q_F B(H_m) Q_F$ ($F \subseteq \Lambda$, F is finite). The result now easily follows.

■

Theorem 6 *Suppose $\aleph_0 \leq k \leq m$, and $\mathcal{A}$ is m-generated with a generating set $\mathcal{G}$ with $Card\mathcal{G} \leq m$. The following are equivalent.*

1. *$\mathcal{A}$ is $R_{<k}D$.*

2. *There is a faithful unital $*$-homomorphism $\rho : \mathcal{A}^+ \to B(H_m)$ such that, for every $\varepsilon > 0$, every finite subset $E \subseteq H_m$ and every finite subset $W \subseteq \mathcal{G}$, there is a projection $P \in B(H_m)$ and a unital $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{M}_P = PB(H_m)P$ such that, for every $e \in E$ and every $a \in W$ we have*
$$\|[\tau(a) - \rho(a)]e\| < \varepsilon.$$

3. *There is a faithful unital representation $\rho : \mathcal{A}^+ \to B(H_m)$ and a net $\{P_\alpha\}$ of projections in $B(H_m)$, each with rank less than k, such that $P_\alpha \to 1$ ($*$-SOT) and such that, for each α, there is a representation $\pi_\alpha : \mathcal{A} \to \mathcal{M}_{P_\alpha}$ such that, for every $a \in \mathcal{A}$, we have*
$$\pi_\alpha(a) \to \rho(a) \ \ (*\text{-SOT}).$$

4. *For every unital representation $\rho : \mathcal{A}^+ \to B(H_m)$ there is a net $\{P_\alpha\}$ of projections in $B(H_m)$, each with rank less than k, such that $P_\alpha \to 1$ ($*$-SOT) and such that, for each α, there is a representation $\pi_\alpha : \mathcal{A} \to \mathcal{M}_{P_\alpha}$ such that, for every $a \in \mathcal{A}$, we have*
$$\pi_\alpha(a) \to \rho(a) \ \ (*\text{-SOT}).$$

Proof. (2) $\Longrightarrow$ (1) Let A be the set of triples (ε, E, W) ordered by $(\geq, \subseteq, \subseteq)$. If $\alpha = (\varepsilon, E, W)$ let $\tau_\alpha : \mathcal{A} \to P_\alpha B(H_m) P_\alpha$ guaranteed by (2). Since $\mathcal{G} = \mathcal{G}^*$ we have
$$(*\text{-SOT})\lim_\alpha \tau_\alpha(a) = \rho(a)$$
for every $a \in \mathcal{G}$. Since ρ and each τ_α is a $*$-homomorphism, the set of $a \in \mathcal{A}$ for which ($*$-SOT)$\lim_\alpha \tau_\alpha(a) = \rho(a)$ is a unital C*-algebra and is thus $\mathcal{A}^+$. Hence, for every $a \in \mathcal{A}^+$, we have
$$\|a\| = \|\rho(a)\| \leq \sup\{\|\tau_\alpha(a)\| : \alpha \in A\}.$$
Therefore the direct sum of the τ_α's is faithful and (1) is proved.

(3) $\Longrightarrow$ (2). This is obvious.

(4) $\Longrightarrow$ (3). It is clear that we need only show that there is a faithful unital representation $\rho : \mathcal{A}^+ \to B(H_m)$. Suppose $\tau : \mathcal{A}^+ \to B(M)$ is an irreducible representation, and suppose D is a generating set with $Card(D) \leq m$. Let $\mathcal{A}_0$ be the unital $*$-subalgebra of $\mathcal{A}^+$ over the field $\mathbb{Q} + i\mathbb{Q}$ of complex rational numbers. Then $\mathcal{A}_0$ is norm dense in $\mathcal{A}$ and $Card\mathcal{A}_0 = CardD \leq m$. Suppose $f \in M$ is a unit vector. Since τ is irreducible, $\tau(\mathcal{A}_0) f$ must be dense in M. Suppose B is an orthonormal basis for M, and, for each $e \in B$ let U_e be the

open ball centered at e with radius $\sqrt{2}/2$. Each U_e must intersect the dense set $\tau(\mathcal{A}_0) f$, and since the collection $\{U_e : e \in B\}$ is disjoint, we conclude that

$$\dim M = CardB \leq Card\tau(\mathcal{A}_0) f \leq Card(\mathcal{A}_0) \leq m.$$

We know that for every $x \in \mathcal{A}_0$ there is an irreducible representation $\tau_x : \mathcal{A}^+ \to B(M_x)$ such that $\|\tau_x(x)\| = \|x\|$. Since $\dim \sum_{x\in\mathcal{A}_0}^{\oplus} M_x \leq m \cdot m = m$, there is a representation $\rho : \mathcal{A}^+ \to B(H_m)$ that is unitarily to a direct sum of m copies of $\sum_{x\in\mathcal{A}_0}^{\oplus} \tau_x$. Hence ρ is isometric on the dense subset $\mathcal{A}_0$, which implies ρ is faithful.

(1) $\Longrightarrow$ (3). Since $\mathcal{A}$ is $R_{<k}D$, we can choose a decomposition $H_m = \sum_{\lambda\in\Lambda}^{\oplus} X_\lambda$ and representation $\pi = \sum_{\lambda\in\Lambda}^{\oplus} \pi_\lambda$ as in part (1) of Lemma 4. Now (3) follows from Lemma 5. ■

We see that the class of $R_{<k}D$ algebras is closed under arbitrary free products in the nonunital category of C*-algebras.

Theorem 7 *Suppose k is an infinite cardinal and $\{\mathcal{A}_\iota : i \in I\}$ is a family of $R_{<k}D$ C*-algebras. Then the free product $\ast_{i\in I} \mathcal{A}_i$ is $R_{<k}D$.*

Proof. Choose an infinite cardinal $m \geq k + \sum_{i\in I} Card(\mathcal{A}_i)$. Since $\ast_{i\in I} \mathcal{A}_i$ is generated by $\mathcal{G} = \left[\bigcup_{i\in I} \mathcal{A}_i\right] \setminus \{0\} \subseteq \ast_{i\in I} \mathcal{A}_i$, clearly $\ast_{i\in I} \mathcal{A}_i$ is m-generated. Choose a set Λ with $Card(\Lambda) = m$ and let S be the set of cardinals less than k. Write

$$H_m = \sum\nolimits_{s\in S}^{\oplus} \sum\nolimits_{\lambda\in\Lambda} X_{s,\lambda}$$

where $\dim X_{s,\lambda} = s$ for every $s \in S$ and $\lambda \in \Lambda$. It follows that, for each $i \in I$, we can find a representation $\pi^i : \mathcal{A}_i \to B(H_m)$ such that

$$\pi^i = \sum\nolimits_{s\in S}^{\oplus} \sum\nolimits_{\lambda\in\Lambda} \pi^i_{s,\lambda}$$

satisfying (1) and (2) of Lemma 4. Suppose $\varepsilon > 0$, $E \subseteq H_m$ is finite and $W \subseteq \mathcal{G}$ is finite. We can write W as a disjoint union of $W_{i_1}, \ldots, W_{i_n}$ with $W_i = W \cap \mathcal{A}_i$. Let ρ_i be the restriction of ρ to $\mathcal{A}_i$. Applying Lemma 5 to $\mathcal{A}_{i_j}$ and ρ_{i_j} and π^{i_j} for $1 \leq j \leq n$, we can find one finite subset $G \subseteq S \times \Lambda$ so that if P is the projection on $\sum\nolimits_{(s,\lambda)\in G}^{\oplus} X_{s,\lambda}$, then there are unitary operators

$U_{i_1}, \ldots, U_{i_n} \in \mathcal{M}_P = PB(H_m)P$ so that, for $1 \leq j \leq n$, $a \in W_j$, $e \in E$, we have

$$\left\| \left[\rho_{i_j}(a) - U_{ij}^* \pi^{i_j}(a) U_{ij} \right] e \right\| < \varepsilon.$$

Define $\tau_{ij} : \mathcal{A}_{i_j}^+ \to \mathcal{M}_P$ by

$$\tau_{ij}(a) = U_{ij}^* \pi^{i_j}(a) U_{ij},$$

and for $i \in I \backslash \{i_1, \ldots, i_n\}$ define $\tau_i : \mathcal{A}_i \to \mathcal{M}_P$ by

$$\tau_i(a) = P\pi^i(a) P.$$

Then, by the definition of free product, there is a representation $\tau : \underset{i \in I}{*} \mathcal{A}_i^+ \to \mathcal{M}_P$ such that $\tau | \mathcal{A}_i = \tau_i$ for every $i \in I$. It follows that, for every $e \in E$ and every $a \in W$,

$$\| [\rho(a) - \tau(a)] e \| < \varepsilon.$$

It follows from part (2) of Lemma 6 that $\underset{i \in I}{*} \mathcal{A}_i$ is $R_{<k}D$. ■

Corollary 8 *Suppose k is an infinite cardinal and $\{\mathcal{A}_\iota : i \in I\}$ is a family of $R_{<k}D$ C^*-algebras such that each $\mathcal{A}_i$ has a one-dimensional unital representation. Then the unital free product $\underset{i \in I}{*_{\mathbb{C}}} \mathcal{A}_i$ is $R_{<k}D$.*

Proof. This follows from the fact that if $\tau_i : \mathcal{A}_i \to \mathbb{C}$ is a unital $*$-homomorphism for each $i \in I$, then $\underset{i \in I}{*_{\mathbb{C}}} \mathcal{A}_i$ is $*$-isomorphic to $\left(\underset{i \in I}{*} \ker \tau_i \right)^+$. ■

Without the condition on unital one-dimensional representations, the preceding corollary is false. For example, $\underset{n \in \mathbb{N}}{*_{\mathbb{C}}} \mathcal{M}_n(\mathbb{C})$ is not RFD ($= R_{<\aleph_0}D$), even though each $\mathcal{M}_n(\mathbb{C})$ is RFD. The reason is that each unital representation of the free product must be injective on each $\mathcal{M}_n(\mathbb{C})$ and must have infinite-dimensional range. call an infinite cardinal k a *limit cardinal*, if k is the supremum of all the cardinals less than k.

However, there is something we can say about the general situation. If k is a limit cardinal, the *cofinality* of k is the smallest cardinal s for which there is a set E of cardinals less than k whose supremum is k. Clearly, the cofinality of k is at most k. If k is not a limit cardinal, then there is a cardinal s such that k is the smallest cardinal larger than s, and if E is a set of cardinals less than k, then $\sup(E) \leq s < k$.

Theorem 9 *Suppose k is an infinite cardinal and $\{\mathcal{A}_\iota : i \in I\}$ is a family of unital $R_{<k}D$ C^*-algebras. Then*

1. *If k is a limit cardinal and $Card(I)$ is less than the cofinality of k, then the free product $\underset{i \in I}{*_{\mathbb{C}}} \mathcal{A}_i$ is $R_{<k}D$.*

2. *If k is not a limit cardinal, then the free product $\underset{i\in I}{*_{\mathbb{C}}}\mathcal{A}_i$ is $R_{<k}D$.*

Proof. (1). Choose $m \geq k+\sum_{i\in I} Card(\mathcal{A}_i)$, and choose a set Λ with $Card(\Lambda) = m$. Using Lemma **??** we can, for each $i \in I$, find a faithful representation $\pi^i = \sum_{\lambda\in\Lambda}\pi_{\lambda,i}$ so that $\dim \pi^i = m$ and, for every $i \in I$ and $\lambda \in \Lambda$, we have $\dim \pi_{\lambda,i} < k$. Since $Card(I)$ is less than the cofinality of k, we have, for each $\lambda \in \Lambda$, a cardinal $s_\lambda < k$ such that $\sup_{i\in I}\dim \pi_{\lambda,i} \leq s_\lambda$. If we replace each $\pi_{\lambda,i}$ with a direct sum of s_λ copies of itself, we get a new decomposition which we will denote by the same names such that, for each i and each λ we have $\dim \pi_{\lambda,i} = s_\lambda$. Hence we may write direct sum decompositions of the π^i's with respect to a common decomposition $H_m = \sum_{\lambda\in\Lambda} X_\lambda$ where $\dim X_\lambda = s_\lambda$ for every $\lambda \in \Lambda$. The rest now follows as in the proof of Theorem 7.

(2) If k is not a limit cardinal, there is a largest cardinal $s < k$. Repeat the proof of part (1) with $s_\lambda = s$ for every $\lambda \in \Lambda$. ■

Remark 10 *We cannot remove the condition on $Card(I)$ in part (1) of Theorem **??**. Suppose k is a limit cardinal and I is a set of cardinals less than k whose cardinality equals the cofinality of k and such that $\sup(I) = k$. For each infinite cardinal m, choose a set Λ_m with cardinality m, and let $\mathcal{S}_m$ denote the universal unital C^*-algebra generated by $\{v_\lambda : \lambda \in \Lambda_m\}$ with the conditions*

1. *$v_\lambda^* v_\lambda = 1$ for every $\lambda \in \Lambda_m$,*

2. *$v_{\lambda_1} v_{\lambda_1}^* v_{\lambda_2} v_{\lambda_2}^* = 0$ for $\lambda_1 \neq \lambda_2$ in Λ_m.*

Since $\mathcal{S}_m$ is m-generated, it follows that every irreducible representation of $\mathcal{S}_m$ is at most m-dimensional (see the proof of $(4) \Longrightarrow (3)$ in Theorem 6). Hence $\mathcal{S}_m$ is separated by m-dimensional representations. On the other hand, if π is a unital representation of $\mathcal{S}_m$, then $\{\pi(v_{\lambda;} v_\lambda^*) : \lambda \in \Lambda_m\}$ is an orthogonal family of nonzero projections, which implies that the dimension of π is at least m.It follows that each $\mathcal{S}_s$ is $R_{<k}D$ for $s \in I$. However, any unital representation π of the free product $\underset{s\in I}{*_{\mathbb{C}}}\mathcal{S}_s$ must induce a unital representation of each $\mathcal{S}_s$, so its dimension is at least $\sup I = k$. Hence $\underset{s\in I}{*_{\mathbb{C}}}\mathcal{S}_s$ is not $R_{<k}D$.

3 Separable RFD Algebras

In this section we show that for a separable C*-algebra being RFD is equivalent to a lifting property.

Suppose $\{e_1, e_2, \ldots\}$ is an orthonormal basis for a Hilbert space ℓ^2, and, for each integer $n \geq 1$, let P_n be the projection onto $sp(\{e_1, \ldots, e_n\})$. Let $\mathcal{M}_n = P_n B(\ell^2) P_n$ for $n \geq 1$, and, following Lemma 1, let

$$\mathcal{B} = \left\{ \{T_n\} \in \prod_{n=1}^{\infty} \mathcal{M}_n : \exists T \in B(\ell^2) \text{ with } T_n \to T \text{ (} *\text{-SOT)} \right\},$$

and let

$$\mathcal{J} = \{\{T_n\} \in \mathcal{B} : T_n \to 0 \text{ (} *\text{-SOT)}\}.$$

Then, by Lemma 1, we have that $\mathcal{B}$ is a unital C*-algebra, $\mathcal{J}$ is a closed ideal in $\mathcal{B}$ and

$$\pi(\{T_n\}) = (\ *\text{-SOT})\text{-} \lim_{n \to \infty} T_n$$

defines a unital surjective $*$-homomorphism from $\mathcal{B}$ to $B(H)$ whose kernel is $\mathcal{J}$. We can now give our characterization of RFD for separable C*-algebras.

Theorem 11 *Suppose $\mathcal{A}$ is a separable C*-algebra. The following are equivalent.*

1. *$\mathcal{A}$ is RFD*
2. *For every unital $*$-homomorphism $\rho : \mathcal{A}^+ \to B(\ell^2)$ there is a unital $*$-homomorphism $\tau : \mathcal{A}^+ \to \mathcal{B}$ such that $\pi \circ \tau = \rho$.*

Proof. The implication (2) $\Longrightarrow$ (1) is clear.

(1) $\Longrightarrow$ (2). Suppose $\mathcal{A} = C^*(\{a_1, a_2, \ldots\})$ is RFD and $\rho : \mathcal{A}^+ \to B(\ell^2)$ is a unital $*$-homomorphism. It follows from Theorem 6 that there is an increasing sequence $\{n_k\}$ of positive integers and unital $*$-homomorphisms $\tau_k : \mathcal{A} \to \mathcal{M}_{n_k}$ such that

$$\|[\tau_k(a_j) - \rho(a_j)] e_i\| < 1/k$$

for $1 \leq i, j \leq k$. It follows that $\tau_{n_k}(a) \to \rho(a)$ ($*$-SOT) for every $a \in \mathcal{A}^+$. If $n_k < n < n_{k+1}$ we define $\tau_n : \mathcal{A}^+ \to \mathcal{M}_n$ by

$$\tau_n(a) = \begin{pmatrix} \tau_{n_k}(a) & & & \\ & \beta(a) & & \\ & & \ddots & \\ & & & \beta(a) \end{pmatrix},$$

where $\beta : \mathcal{A}^+ \to \mathbb{C}$ is the unique $*$-homomorphism with $\ker \beta = \mathcal{A}$, relative to the decomposition

$$P_n(\ell^2) = P_{n_k}(\ell^2) \oplus \mathbb{C}e_{1+n_k} \oplus \cdots \oplus \mathbb{C}e_{-1+n_{k+1}}.$$

It is easily seen that $\tau_n(a) \to \rho(a)$ ($*$-SOT) for every $a \in \mathcal{A}^+$. If we define $\tau : \mathcal{A} \to \mathcal{B}$ by

$$\tau(a) = \{\tau_n(a)\},$$

we see that $\pi \circ \tau = \rho$. ■

Acknowledgement 12 *The author wishes to thank Tatiana Shulman and Terry Loring for bringing the question answered by Theorem 11 to his attention. Thinking about this question led to the discovery of all the results in this paper.*

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